Statistical mechanics far from equilibrium

1969 J. Phys. A: Gen. Phys. 2 157

(http://iopscience.iop.org/0022-3689/2/2/003)

J. PHYS. A (GEN. PHYS.), 1969, SER. 2, VOL. 2. PRINTED IN GREAT BRITAIN

Statistical mechanics far from equilibrium

S. F. EDWARDS and W. D. McCOMB

Department of Theoretical Physics, Schuster Laboratory, University of Manchester

MS. received 8*th November* 1968

Abstract. A study is made of the statistical mechanics of systems in which the dominant process is a flow of energy through the modes of the system. The case of randomly excited fluid turbulence is studied in detail and it is argued that there is a strong mathematical analogy between the classical (turbulent) cascade of energy and the quantum field or many-body problem. The energy has an analogue in the one-particle Green function, and entropy can be defined, the latter being the information content in the case of the probability distribution function for the turbulence. It is further argued that general operations in Hilbert space can be carried out with at most two functions, and that the energy equation and the maximization of the entropy give two equations which determine the two chosen unknown functions. As an example, the case of a random long-wave imput of energy is studied and shown to lead to the Kolmogoroff spectrum, and the Kolmogoroff constant is evaluated for the approximation system used.

1. Introduction

A variety of physical problems involve systems with a very large or infinite number of degrees of freedom. The many-body problem in quantum and classical statistical mechanics is perhaps, currently, the most studied, but the quantum field theory of elementary particles is another evidently related case, whereas the statistical theory of turbulence and the theory of random systems are important problems which have not received so much attention in their theoretical aspects; a general account of such problems is given in the book by Beran (1968). In this paper, a general discussion of these problems will be given, but as a detailed example calculations will be given for the case of turbulence, since this presents the full problem in its simplest form. This is not to say that turbulence is at all simple, but one cannot argue away from perturbation theory etc., as in the other problems, and one has to face the full problem at once. It is possible to cast these physical problems in the form of differential equations in function space, and it is the solution of these equations which will concern this paper. For example, consider a fluid which satisfies the Navier–Stokes equations with a body force $\mathscr{F}$. If we let $U^\alpha(\boldsymbol{r})$ be the velocity field, then

$$\frac{\partial \boldsymbol{U}}{\partial t} = \nu \nabla^2 \boldsymbol{U} - (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} - \nabla p + \mathscr{F} \tag{1.1}$$

$$\nabla \cdot \boldsymbol{U} = 0 \tag{1.2}$$

ν being the viscosity and p the pressure, determined by the incompressibility condition (1.2). In terms of the Fourier components in a box of side L:

$$\boldsymbol{U}(\boldsymbol{r}, t) = L^{-3} \sum_{\boldsymbol{k}} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{U}_{\boldsymbol{k}}(t) \tag{1.3}$$

where $\boldsymbol{k} = (2\pi/L)(n_1, n_2, n_3)$, n integers, one can write (1.1) as

$$\frac{\partial U_{\boldsymbol{k}}{}^\alpha}{\partial t} = -\nu k^2 U_{\boldsymbol{k}}{}^\alpha + \sum_{j,l;\beta\gamma} M_{\boldsymbol{k},j,l}^{\alpha\beta\gamma} U_j{}^\gamma U_l{}^\beta + \sum_\beta \mathscr{F}_{\boldsymbol{k}}{}^\beta \mathscr{D}_{\boldsymbol{k}}{}^{\alpha\beta} \tag{1.4}$$

where $k^\alpha U_{\boldsymbol{k}}{}^\alpha = 0$, a condition which has been used to eliminate the pressure with the resulting introduction of M and $\mathscr{D}$.

$$\mathscr{D}_{\boldsymbol{k}}{}^{\alpha\beta} = \delta^{\alpha\beta} - k^\alpha k^\beta |\boldsymbol{k}|^{-2} \tag{1.5}$$

$$M_{kjl}^{\alpha\beta\gamma} = \frac{\Delta}{\mathrm{i}(2\pi)^3} (k^\beta \mathscr{D}_{\boldsymbol{k}}{}^{\alpha\gamma} + k^\gamma \mathscr{D}_{\boldsymbol{k}}{}^{\alpha\beta}) \delta_{kjl} \tag{1.6}$$

where $\Delta = (2\pi/L)^3$ and $\delta_{kjl} = 1$ if $\boldsymbol{k}+\boldsymbol{j}+\boldsymbol{l} = 0$, zero otherwise. In the limit of $L \to \infty$, Δ becomes $\mathrm{d}^3\boldsymbol{k}$ and

$$\delta_{kjl} \to \Delta\delta(\boldsymbol{k}+\boldsymbol{j}+\boldsymbol{l}).$$

Now suppose that the force $\mathscr{F}$ fluctuates randomly in time with a Gaussian distribution

$$\mathscr{P}(\mathscr{F}) = \mathscr{N} \exp\left\{-\int \frac{\mathscr{F}_k{}^{\alpha}(t)\mathscr{F}_k{}^{\beta}(t)\,\mathrm{d}t}{h_k}\right\} \tag{1.7}$$

where $\mathscr{N}$ is a normalization $\int\mathscr{P}\delta\mathscr{F} = 1$, where $\delta\mathscr{F}$ means integration over all functions so that if $\langle\rangle$ means average value

$$\langle\mathscr{F}_k{}^{\alpha}(t)\mathscr{F}_k{}^{\beta}(t')\rangle = 2h_k\delta(t-t')\delta^{\alpha\beta}. \tag{1.8}$$

Now let us consider the probability $P(\ldots u_k \ldots t)$ of finding the $U_k{}^{\alpha}$ to have the values $u_k{}^{\alpha}$ at the time t. The Liouville equation will be satisfied:

$$\frac{\partial P}{\partial t} = \sum_k \frac{\partial}{\partial u_k{}^{\alpha}}\left(-\nu k^2 u_k{}^{\alpha} + \sum_{j,l;\beta\gamma} M_{kjl}^{\alpha\beta\gamma} u_j{}^{\beta}u_l{}^{\gamma} + \sum_{\beta} \mathscr{F}_k{}^{\beta}\mathscr{D}_k{}^{\alpha\beta}\right) P. \tag{1.9}$$

It is a well known theorem that if one denotes $\int P\mathscr{P}(\mathscr{F})\delta\mathscr{F}$ by $\langle P\rangle$, then

$$\frac{\partial\langle P\rangle}{\partial t} + \sum_k \frac{\partial}{\partial u_k{}^{\alpha}}\left(\nu k^2 u_k{}^{\alpha} - \sum_{j,l;\beta\gamma} M_{kjl}^{\alpha\beta\gamma} u_j{}^{\beta}u_l{}^{\gamma} + h_k \frac{\partial}{\partial u_{-k}{}^{\alpha}}\right)\langle P\rangle = 0. \tag{1.10}$$

(A proof is given by Chandrasekhar (1943) and also, in our present notation and point of view, by Edwards (1964, to be referred to as I).) Since (1.10) will be a basic equation of this paper and only $\langle P\rangle$ will appear in future the brackets will be dropped and P used henceforward for the solution of (1.10).

The energy balance equation is obtained by multiplying (1.10) by the energy $\frac{1}{2}\int u^2(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}$, i.e. $\Sigma_k u_k u_{-k}$, which one can do for each component separately. Let

$$\mathscr{D}_k{}^{\alpha\beta} q_k = \int u_k{}^{\alpha}u_k{}^{\beta} P\delta\boldsymbol{u}$$

then

$$\frac{1}{2}\frac{\partial q_k}{\partial t} + \nu k^2 q_k + \int \sum_{j,l} M_{kjl}^{\alpha\beta\gamma} u_j{}^{\beta}u_l{}^{\gamma}u_k{}^{\alpha} P\delta\boldsymbol{u} = h_k. \tag{1.11}$$

The total energy $E = \frac{1}{2}\int q_k\,\mathrm{d}^3\boldsymbol{k}$, and, as the M term vanishes upon integration over $\boldsymbol{k}$,

$$\frac{\partial E}{\partial t} + \int \nu k^2 q_k\,\mathrm{d}^3\boldsymbol{k} = \int h_k\,\mathrm{d}^3\boldsymbol{k}. \tag{1.12}$$

Equation (1.11) means in words: for each mode $\boldsymbol{k}$ (change in energy) + (loss due to viscosity) + (transfer into and out of other modes) = (gain from outside). Equation (1.12) means: (total change in energy) + (total loss due to viscosity) = (total input from outside). Physically one can think of h_k as the rate of input of energy due to some random stirring force; the energy cascades through the system via the Muu term and disappears at large $\boldsymbol{k}$ owing to the viscosity. Provided $h_k = 0$ for $|\boldsymbol{k}| <$ some K_1, and $\int h_k\,\mathrm{d}^3\boldsymbol{k} = h$, which is finite, one can expect a steady, if somewhat idealized, state to build up and a P which is time independent to result where

$$\sum_{k,\alpha} \frac{\partial}{\partial u_k{}^{\alpha}}\left(\nu k^2 u_k{}^{\alpha} - \sum_{j,l;\beta\gamma} M_{kjl}^{\alpha\beta\gamma} u_j{}^{\beta}u_l{}^{\gamma}\right) P + \sum_{k,\alpha} h_k \frac{\partial^2}{\partial u_k{}^{\alpha}\,\partial u_{-k}{}^{\alpha}} P = 0. \tag{1.13}$$

This seems to be the simplest formulation of a turbulence problem: a differential equation in an infinite set of variables u_k where $\boldsymbol{k}$ fills a Cartesian three-dimensional space. No difficulty seems occasioned by letting the discrete $\boldsymbol{k}$ tend to the full infinite three-dimensional

continuum, and indeed it makes the mathematics simpler to do so. It is worth remarking that most work on turbulence discusses the much more difficult problem of the time correlations $\langle U_k(t)U_{-k}(0)\rangle$, but since there is a differential equation for the instantaneous correlation alone, this only will be in this paper apart from comments.

Quantum field theory and the quantum many-body problem can be expressed in a similar way. Physical quantities can be evaluated from Green functions which are the correlation functions:

$$\langle\psi(\boldsymbol{r}_1, t_1)\psi(\boldsymbol{r}_2, t_2)\ldots\psi^*(\boldsymbol{r}_3, t_3)\psi^*(\boldsymbol{r}_4, t_4)\rangle = \int \psi(\boldsymbol{r}_1, t_1)\ldots\psi^*(\boldsymbol{r}_3, t_3)\ldots e^{(i/\hbar)S}\delta\psi\delta\psi^* \quad (1.14)$$

where S is Hamilton's principal function, i.e. the time integral of the Lagrangian. For example, a system of electrons interacting with potential $V(\boldsymbol{r})$ has, in the usual notation,

$$S = \int d^3x\, dt(\psi\dot{\psi}^* - \psi^*\dot{\psi} - \frac{\hbar^2}{2m}(\nabla\psi)(\nabla\psi^*) - \mu\psi\psi^*)$$

$$+ \int V(\boldsymbol{r}-\boldsymbol{r}')\psi(\boldsymbol{r})\psi^*(\boldsymbol{r})\psi(\boldsymbol{r}')\psi^*(\boldsymbol{r}')\, d^3\boldsymbol{r}\, d^3\boldsymbol{r}'\, dt. \quad (1.15)$$

Now one can put

$$e^{iS/\hbar} = P \quad (1.16)$$

and see that

$$\frac{\hbar}{i}\frac{\partial P}{\partial\psi} = \left(\frac{\partial S}{\partial\psi}\right)P \quad (1.17)$$

$$\frac{\hbar}{i}\frac{\partial P}{\partial\psi^*} = \left(\frac{\partial S}{\partial\psi^*}\right)P \quad (1.18)$$

so that to obtain a form like (1.13) one may differentiate again to get

$$\left(\frac{\hbar}{i}\frac{\partial^2}{\partial\psi\partial\psi^*} - \frac{\partial}{\partial\psi}\frac{\partial S}{\partial\psi^*}\right)P - (\text{complex conjugate}) = 0. \quad (1.19)$$

Of course one can go ahead and solve this equation back to (1.15), whereas one cannot solve (1.13) in any simple way. However, the existence of an exact and explicit form (1.15) still seems to be of little help in actually obtaining the values of $\langle\psi(\boldsymbol{r}_1,t_1)\ldots\psi^*\ldots\rangle$ so the differential form may well be a better starting point (see for example Edwards and Sherrington (1967) who develop the many-body problem from this point of view). Problems of disordered systems can also be cast in similar forms to (1.13), but we shall not pursue that here.

When one considers non-linear differential equations of an infinite order it is clear that only rather simple and systematic operations can be usefully carried out in the Hilbert space spanned by the equation. One is led to ask what is the most general operation possible in such a space. Clearly linear operations are straightforward, but the most general operation possible on non-linear forms is described by the theorem which states that any two symmetric or Hermitian forms may be simultaneously diagonalized. This theorem cannot be generalized in any way to more than two forms, or to cubic and higher forms. This theorem is reflected in the answer to the following question. If some model is taken for the differential equation (1.13) (or (1.19)) and then used in an expansion theorem, what is the most general differential operator in a Hilbert space which permits such an expansion, i.e. if one takes

$$(\mathscr{L}_0+\mathscr{L}_1)f = 0$$

$$f = f_0 - \frac{1}{\mathscr{L}_0}\mathscr{L}_1 f_0 + \frac{1}{\mathscr{L}_0}\mathscr{L}_1\frac{1}{\mathscr{L}_0}\mathscr{L}_1 f_0 + \ldots \quad (1.20)$$

what is the most general $\mathscr{L}_0$ possible in a space of an infinite number of dimensions? The operator $\mathscr{L}_0$ must contain two and not more than two quadratic forms, and be invertible: it must be a form of Hermite's operator

$$\left\{\sum_{i,j} a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\sum_{i,j} b_{ij}x_i x_j\right\}=\mathscr{L}_0. \tag{1.21}$$

(Linear terms in $\partial/\partial x$ and x may be added, but are trivial.) The theorem permits a transformation

$$x_i = K_{ij}y_j$$

so that

$$\mathscr{L}_0=\sum_\alpha A_\alpha\frac{\partial^2}{\partial y_\alpha{}^2}+\sum_\alpha B_\alpha y_\alpha{}^2. \tag{1.22}$$

In the case of (1.13) one has chosen a problem which is already homogeneous and isotropic, so the argument now is that the simplest model for (1.13) is

$$\sum_k\left(D_k\frac{\partial^2}{\partial u_k\partial u_{-k}}+\omega_k\frac{\partial}{\partial u_k}u_{-k}\right)P_0=0 \tag{1.23}$$

$$P_0=\prod_k\left(\frac{D_k}{\omega_k}\right)^{-3/2}\exp\left\{\int u_k u_{-k}\frac{\omega_k}{D_k}\mathrm{d}^3\boldsymbol{k}\right\}. \tag{1.24}$$

That this happens to have a similar form to the case without non-linearity is coincidental from the mathematical point of view, as is the fact that the system is already diagonal. (For pipe flow, for example, it would not be so.) The coincidence, however, indicates that the mathematics is telling one that the concepts of a generalized diffusivity D_k and a generalized viscosity (or dynamical friction) ω_k are fruitful in the sense that they are mathematically tractable. It is important to emphasize again that it is not possible to construct a model with more than two unknown functions and still be able to use it generally, and equally important to realize that a model with only one unknown would not be using the full power of the mathematics available to us. One must now find criteria, both physical and mathematical, for the choice of D_k, ω_k and hence solve the problem.

2. The energy expansion

Firstly consider (1.11) from the point of view of its physical content. There is an analogy between (1.13) and (1.19), as has been pointed out, and the analogy is heightened by the fact that whereas $\langle u_k u_{-k}\rangle$ represents the energy in the mode $\boldsymbol{k}$, $f=\langle\psi_k(t)\psi_k{}^*(t)\rangle$ represents the number of electrons in the state k, i.e. of velocity $\hbar\boldsymbol{k}/m$. The total number of particles

$$N=\int f(\boldsymbol{v})\,\mathrm{d}^3\boldsymbol{v} \tag{2.1}$$

whereas the total energy

$$E=\tfrac{1}{2}\int g_k\,\mathrm{d}^3\boldsymbol{k}. \tag{2.2}$$

Now the distribution of particles satisfies the Boltzmann equation under the appropriate conditions (the Peierls–Boltzmann equation as applied to phonons, say, in the problem of thermal conductivity is the nearest analogy (Peierls 1955), which has the form

$$\left(\frac{\partial f}{\partial t}\right)+\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}}+\int w(\boldsymbol{v}\boldsymbol{v}'\boldsymbol{v}_1\boldsymbol{v}_1')\{f(\boldsymbol{v})f(\boldsymbol{v}_1)-f(\boldsymbol{v}')f(\boldsymbol{v}_1')\}\,\mathrm{d}(\boldsymbol{v}_1\ldots\boldsymbol{v}_1')$$
$$=\text{sources}+\text{sinks} \tag{2.3}$$

where w is the scattering probability. There is of course no justification for believing this simplest form of transport equation will be valid, but nevertheless, if it were, one could expect

$$\left(\frac{\partial q_k}{2\partial t}\right)+\nu k^2 q_k+\int \Lambda_{kjl}q_j q_l-\int \Lambda_{ljk}q_k q_l = h_k \tag{2.4}$$

where $\partial q_k/2\partial t$ is put in brackets since our discussion has been for the case of a steady state. If, however, external conditions h_k, νk^2 changed very slowly in time, such a $(\partial q_k/2\partial t)$ could be expected. The two terms Λqq represent the rate at which energy flows into the mode q_k, and out of it. If these two terms had been more generally defined the equation would have been a Pauli master equation, but it has already been cast in this form since the differential equation clearly suggests that the simplest flow of energy will be between the triad $\boldsymbol{k}, \boldsymbol{j}, \boldsymbol{l}$. As is known from modern discussions of the extensions of the Peierls–Boltzmann equation, one can expect corrections to the kernel like $\Lambda^{(1)}q_k q_j q_l$, etc., or, alternatively, one can say that it can be expected that if form (2.4) is exact, then the Λ can be expected to be quite complicated. If one followed Peierls' original derivation of his equation and thus calculated Λ in perturbation theory one would find

$$\Lambda_{kjl} = \frac{L_{kjl}}{\nu k^2+\nu j^2+\nu l^2} \tag{2.5}$$

and

$$L_{kjl} = \tfrac{1}{2}\delta_{kjl}M_{kjl}^{\alpha\beta\gamma}M_{lkj}^{\alpha\beta'\gamma'}(\mathscr{D}_j^{\beta\beta'}\mathscr{D}_l^{\gamma\gamma'}+\mathscr{D}_j^{\beta\gamma'}\mathscr{D}_j^{\gamma\beta'}) \tag{2.6}$$

the essential difference being that the Peierls case has real scattering and so $\delta(\epsilon_k-\epsilon_j-\epsilon_l)$ where here one has dissipation, and hence $(\nu k^2+\nu j^2+\nu l^2)^{-1}$ where ϵ_k is the energy of the particle labelled k, i.e.

$$\int_{-\infty}^{\infty} \exp\{i\tau(\epsilon_k-\epsilon_j-\epsilon_l)\}\,d\tau = 2\pi\delta(\epsilon_k-\epsilon_j-\epsilon_l) \tag{2.7}$$

$$\int_{0}^{\infty} \exp\{-\tau(\nu k^2+\nu j^2+\nu l^2)\}\,d\tau = \frac{1}{\nu k^2+\nu j^2+\nu l^2}. \tag{2.8}$$

(An important application of the Peierls–Boltzmann equation has been made by Hasselmann to the interaction of water waves which leads us to a generalization of (2.3).) The details of this derivation will not be given here since the present object is to get away from perturbation theory altogether, nevertheless it is important to realize that this form of equation has important properties such as the fact that under all conditions $q_k(t) \geqslant 0$ and

$$\int d^3\boldsymbol{k}(\Lambda_{kjl}q_j q_l-\Lambda_{ljk}q_k q_l) = 0.$$

(In the absence of νk^2 and h_k the Boltzmann H theorem is also valid, but that is not useful in the present situation.)

Equation (2.4) has been derived for turbulence in I, and that argument will now be briefly recapitulated. In the form

$$\frac{\partial P}{\partial t}+\sum_k \frac{\partial}{\partial u_k}(\nu k^2+M_{kjl}u_j u_l)P+\sum_k \frac{\partial}{\partial u_k}\mathscr{F}_k^{\beta}\mathscr{D}_k^{\alpha\beta}P = 0$$

the averaging over $\mathscr{F}$ leads to the replacement of $(\partial/\partial u)(\mathscr{F})$ by $h_k\partial^2/\partial u_k\partial u_{-k}$ where $h_k = \int\langle\mathscr{F}\mathscr{F}\rangle\,dt$. It can be argued therefore that if ΣMuu were to be regarded as a random variable one might expect $Muu\,\partial/\partial u$ to be replaced in an equation for an 'average' P by something of order

$$MM\langle uu\rangle\langle uu\rangle\tau$$

where τ has the dimensions of a time. Put explicitly, if one writes (1.10) as

$$\left[\sum_{\alpha,\beta}\left(\frac{\partial}{\partial u_k^{\alpha}}\frac{\partial}{\partial u_k^{\beta}}D_k+\frac{\partial}{\partial u_k^{\alpha}}\omega_k u_k^{\alpha}\right)+\sum_{\alpha,\beta}\left\{\frac{\partial}{\partial u_k^{\alpha}}\frac{\partial}{\partial u_k^{\beta}}(h_k-D_k)+\frac{\partial}{\partial u_k^{\alpha}}u_k^{\alpha}(\nu k^2-\omega_k)\right\}\right.$$

$$\left.+\sum_{\alpha\beta\gamma}M_{kjl}^{\alpha\beta\gamma}u_j^{\beta}u_l^{\gamma}\frac{\partial}{\partial u_k^{\alpha}}\right]P=0 \tag{2.9}$$

and expands, ascribing to h_k-D_k and $\nu k^2-\omega_k$ the normal order M^2, one has

$$P=P_0+P_1+P_2+\ldots \tag{2.10}$$

$$\sum_{\alpha,\beta}\left\{\frac{\partial}{\partial u_k^{\alpha}}\left(\frac{\partial}{\partial u_k^{\beta}}D_k+\omega_k u_k^{\alpha}\right)\right\}P_0=0 \tag{2.11}$$

$$P_0=\prod_k\left(\frac{D_k}{\omega_k}\right)^{-1/2}\exp\left\{\int u_k u_{-k}\left(\frac{\omega_k}{D_k}\right)\mathrm{d}^3\boldsymbol{k}\right\}. \tag{2.12}$$

Since

$$\int P_0 u_k u_{-k}\,\mathrm{d}^3\boldsymbol{k}=q_k \tag{2.13}$$

and since (by the central limit theorem)

$$\int P\prod_{k\neq j}\delta u_j=\prod_k q_k^{-1/2}\exp\left\{-\int\left(\frac{u_k u_{-k}}{q_k}\right)\mathrm{d}^3\boldsymbol{k}\right\} \tag{2.14}$$

we choose that

$$q_k=\frac{D_k}{\omega_k}.$$

Then

$$\sum_{\alpha,\beta}\left\{\frac{\partial}{\partial u_k^{\alpha}}\left(\frac{\partial}{\partial u_k^{\beta}}D_k+\omega_k u_k^{\alpha}\right)\right\}P_1=\sum_{\alpha\beta\gamma}M_{kjl}^{\alpha\beta\gamma}u_j^{\beta}u_l^{\gamma}\frac{\partial}{\partial u_k^{\alpha}}P_0 \tag{2.15}$$

or

$$P_1=\sum_{\substack{\alpha\beta\gamma;\\ jl}}\frac{M_{kjl}^{\alpha\beta\gamma}u_j^{\beta}u_l^{\gamma}(u_k^{\alpha}/q_k)}{\omega_k+\omega_j+\omega_l}P_0 \tag{2.16}$$

$$\sum_{\alpha\beta}\frac{\partial}{\partial u_k^{\alpha}}\left(\frac{\partial}{\partial u_k^{\beta}}D_k+\omega_k u_k^{\alpha}\right)P_2=\sum_{\alpha\beta\gamma}M_{kjl}^{\alpha\beta\gamma}u_j^{\beta}u_l^{\gamma}\frac{\partial P_1}{\partial u_k^{\alpha}}+\sum_{\alpha\beta}\frac{\partial}{\partial u_k^{\alpha}}\left\{\frac{\partial}{\partial u_k^{\beta}}(h_k-D_k)+u_k^{\alpha}(\nu k^2-\omega_k)\right\}P_0 \tag{2.17}$$

and so on.

If indeed (2.13) is to be satisfied one must not have terms in $u_k u_k$ or constants in P_2, which leads to

$$D_k-h_k+q_k(\nu k^2-\omega_k)=\int\int\frac{L_{lkj}q_k q_l-L_{kjl}q_j q_l}{\omega_k+\omega_j+\omega_l}\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l} \tag{2.18}$$

i.e.

$$\nu k^2 q_k+\int\int(\Lambda^0_{lkj}q_k q_l-\Lambda^0_{kjl}q_j q_l)\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}=h_k.$$

The series

$$P = P_0 + P_1 + P_2 + \ldots$$

terminates for the variable $\mathscr{F}$ when averaged, at the P_2 level, but in the general problem it does not, and to order P_2 one has

$$P = \Big\{1 + \sum_{jl} \frac{M_{kjl}u_j u_l u_k/q_k}{\omega_k+\omega_j+\omega_l} + \sum_{\substack{jl'\\ lk'}} \frac{M_{kjl}u_k u_l M_{k'jl'}}{\omega_k+\omega_l+\omega_{k'}+\omega_{l'}} \times \frac{u_{k'}u_{l'}}{q_k} + \sum_{jl;k'l'} \frac{M_{kjl}u_k u_j M_{k'l'l}u_{k'}u_{l'}/q_k}{\omega_k+\omega_j+\omega_{k'}+\omega_{l'}} + \sum_{jlk'j'l'} \frac{M_{kjl}u_j u_l (u_k/q_k) M_{k'j'l'}u_{j'}u_{l'}(u_{k'}/q_k)}{\omega_k+\omega_j+\omega_l+\omega_{k'}+\omega_{j'}+\omega_{l'}} \Big\} P_0. \tag{2.19}$$

A first vital point to notice here is that the one condition (2.13) has produced the energy equation (2.18), but that the relationship between q_k and ω_k is still completely open, and the resolution of the question of how to fix this relationship is the main point of this paper. Secondly one can in principle write down the nth order of the expansion by a diagrammatic technique invented in I. These diagrams are somewhat like the Feynman diagrams, but Feynman diagrams invented firstly for perturbation theory are most easily generalized by replacing perturbation values of $\langle u_k u_{-k}\rangle$ (i.e. of Green functions) by true values which is in the present point of view a 'one quadratic form' theory. The kind of split into two forms used here will require generalization. (There is an extensive discussion of Feynman diagrams and their generalization given by Wyld (1961), which is summarized by Beran (1968).) The generalization appears once a study is made of the series for P. It is clear that the problem encountered is to solve

$$\sum_k \left(\frac{\partial}{\partial u_k} \frac{\partial}{\partial u_{-k}} D_k + \frac{\partial}{\partial u_k} \omega_k u_k \right) P_n = P_0 \times \text{a polynomial in the } u_j.$$

The right-hand side can be rearranged into Hermite polynomials in the u_j, hence into the form

$$\sum_k \frac{\partial}{\partial u_k} \left(\frac{\partial}{\partial u_k} D_k + \omega_k u_k \right) P_n = \sum_{a_k} C^{(n)}_{\ldots a_k a_{-k} \ldots} \prod_k H_{a_k a_{-k}} P_0 \tag{2.20}$$

where a_k denotes the order of the polynomial. For example

$$C^{(1)} = C^{(1)}_{1_k 1_j 1_l} u_k^{\alpha} u_j^{\beta} u_l^{\gamma}, \qquad C^{(1)}_{1_k 1_j 1_l} = M_{kjl}^{\alpha\beta\gamma}. \tag{2.21}$$

The solution is

$$P_n = \sum_{a_k} \frac{C^{(n)}_{\ldots a_k a_{-k} \ldots}}{\sum_{a_k} a_k \omega_k} \prod_k H_{a_k a_{-k}} P_0. \tag{2.22}$$

At this point the limit of infinite volume effects a great simplification for it turns out that only polynomials containing a particular u_k at most once need be kept. Thus one must carry $u_k u_j \ldots (\boldsymbol{k} \neq \boldsymbol{j} \neq \ldots)$ and $u_k u_{-k} - q_k$ but not $u_k{}^2$ or $u_k{}^3$ etc. These all affect the answer to order $1/V$, i.e. $(\mathrm{d}^3\boldsymbol{k})$. The condition (2.13) amounts to saying also that $u_k u_{-k} - q_k$ does not appear either, so that only first powers of any u_k remain in the series for P, and the surviving $C^{(n)}$ can be labelled $C^{(n)}_{\ldots k,j,\ldots}$ without loss of generality. To see the structure of P, one notes that the solution of (2.20) has only the effect of altering the coefficient from

$C^{(n)}$ to $C^{(n)}(\Sigma\omega_k)^{-1}$. This process can be understood in a graphical representation as will be seen, so the expansion is in

$$M_{kjl}u_ju_l\frac{\partial}{\partial u_k} \quad \text{and} \quad \frac{\partial}{\partial u_k}\frac{\partial}{\partial u_{-k}}(D_k-h_k)+\frac{\partial}{\partial u_k}(\omega_k-\nu k^2)u_k$$

and if the factors $(\Sigma_k\omega_k)^{-1}$ are denoted by Ω, the series for P consists of terms like

$$\frac{1}{\Omega}Muu\frac{\partial}{\partial u}\frac{1}{\Omega}Muu\frac{\partial}{\partial u}\frac{1}{\Omega}Muu\frac{\partial}{\partial u}\ldots P_0$$

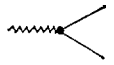
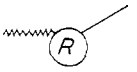
and

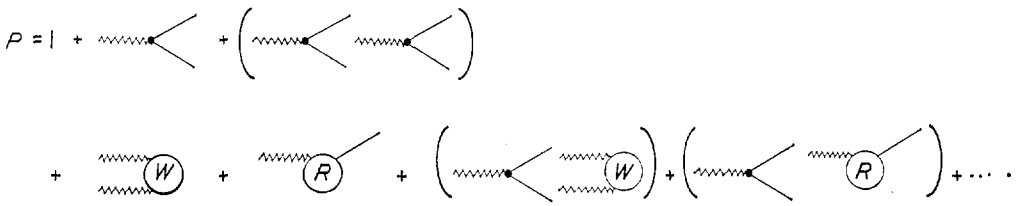
$$\frac{1}{\Omega}\left\{\frac{\partial}{\partial u}\left(\frac{\partial W}{\partial u}+Ru\right)\right\}\frac{1}{\Omega}\left\{\frac{\partial}{\partial u}\left(\frac{\partial W}{\partial u}+Ru\right)\right\}\ldots P_0$$

where

$$W = D_k-h_k \tag{2.23}$$

$$R = \omega_k-\nu k^2. \tag{2.24}$$

(In I, W was called S, but that symbol is required here for the entropy.) Now consider $Muu\ \partial/\partial u$ to be denoted graphically by  , the wavy line for $\partial/\partial u$, full lines for u and M by a dot. Similary the other differential operators are  and  . Since numerical factors Ω^{-1} have to be interposed to translate the graphical series into algebra, the symbols must be kept on a line:



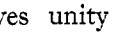
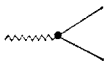
(2.25)

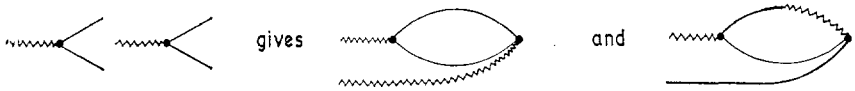
To evaluate any expectation value $\int u_ku_l\ldots P\Pi_k\,du_k$ the $\partial/\partial u$ operators are most conveniently used to operate, by parts, to the left. They then either meet one of the u to their left, or one of the u multiplying P to form the expectation value, or act on unity and give zero. Thus the value of $\int p\delta u$ is given at once by $\int p_0\delta u = 1$, since all other terms commence with a differential and give zero. To use the condition

$$\int u_ku_{-k}P\delta u = \int q_kP\delta u = q_k\int P_0\delta u = q_k \tag{2.26}$$

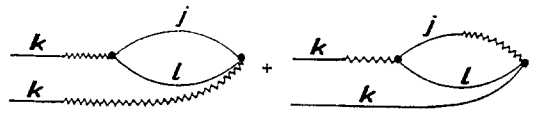
one finds that, since the first term in the expansion of P already gives q_k,

$$\int u_ku_{-k}(P-P_0)\prod_k du_k = 0. \tag{2.27}$$

This integral will select those diagrams which are non-zero when two lines only are allowed to emerge to meet u_k and u_{-k}. All other lines must be paired either in the form which contributes q_k, i.e. $\langle u_ku_{-k}\rangle$ or  which gives unity $\langle u_k\,\partial/\partial u_k\rangle$. Thus  gives zero,



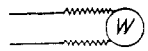
which combine with the $u_k u_{-k}$ to give



These diagrams are completed by insertion of the Ω factor which consists of $(\Sigma\omega)^{-1}$, and ω being inserted for every line between the dots, i.e. $(\omega_k+\omega_j+\omega_l)^{-1}$. Thus their value is

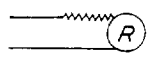
$$\int\int \frac{M_{kjl}M_{lkj}q_l q_j}{\omega_k+\omega_j+\omega_l}\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l} \quad \text{and} \quad \int\int \frac{M_{kjl}M_{jkl}q_k q_l}{\omega_k+\omega_j+\omega_l}\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}.$$

These two diagrams from W and R are likewise



(2.28)

and



(2.29)

i.e. W_k and R_k. Thus the condition (2.27) becomes

$$W_k - R_k q_k + \int\int \mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}\left(\frac{M_{kjl}M_{lkj}q_l q_j - M_{kjl}M_{jkl}q_k q_l}{\omega_k+\omega_j+\omega_l}\right) = 0 \tag{2.30}$$

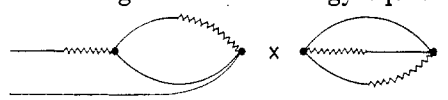
or

$$\nu k^2 q_k + \int \mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}\;\Lambda_{kjl}q_j q_l - \int \mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}\,\Lambda_{ljk}q_k q_l = h_k$$

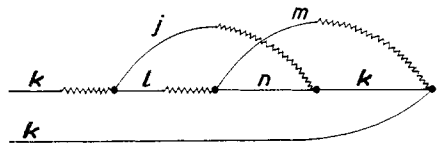
which is our energy balance equation. Higher terms in the series may be computed and those observations which can be made about the conventional expansion methods may still be applied here. For example all the terms in the series must lead to quantities of the order of the volume of the system, which will arise when one takes the limit of a discrete set of $\boldsymbol{k}$ into the continuum

$$\sum_{\boldsymbol{k}} \to \frac{V}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{k}. \tag{2.31}$$

No terms in V^2 can occur on physical grounds, and indeed mathematical proofs of this can also be given. This means that the complete set of diagrams in the energy equation are always *connected*, and one never has things like



For example, to the next order into the series, one gets terms



(2.32)

like

$$\int \frac{\mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{m}\,\mathrm{d}^3\boldsymbol{n}\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}\;MMMMq_n q_p q_k}{(\omega_p+\omega_m+\omega_k)(\omega_m+\omega_n+\omega_k+\omega_j)(\omega_j+\omega_l+\omega_k)} \tag{2.33}$$

and permutations. Although the present diagrams differ from the Feynman-type diagrams in that two types of line appear, and Ω^{-1} factors, the *topology* of the diagrams is the same. It follows that the theorems concerning connectivity of cluster expansions etc. will hold in the present case and need not be repeated.

It is clear that all the higher corrections can be compressed into the notation (2.4), by allowing the Λ to be series in q_k and $(\Sigma\omega)^{-1}$, so that conservation and positive definiteness laws hold to all orders of accuracy.

So the present situation is that the condition (2.27) has led to an energy equation, but this equation contains an unknown function ω_k. Now the assignment of ω_k has to be made.

3. The use of an entropy function

Since the development involved two unknown functions clearly two conditions need to be invoked. In the previous section expanding about the mean value has given one condition which is in fact the energy equation. The question now arises as to how the second condition can occur. Previous approaches to this problem have been made though not perhaps in quite so explicit a form, i.e. in previous treatments the fact that the ω_k really is completely open at this stage was not appreciated. For example, in I, it was argued that since ω_k has the dimension of a lifetime, it should be taken to be the response lifetime of the system to a small disturbance in the external parameters in the mode $\boldsymbol{k}$. Since under a slowly varying change in h_k and νk^2

$$\frac{\partial q_k}{\partial t} = \nu k^2 q_k + \int\int \mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}(\Lambda_{kjl}q_j q_l - \Lambda_{jlk}q_j q_k) - h_k \tag{3.1}$$

if $\delta\nu$ and δh cause δq_k

$$\frac{\partial}{\partial t}\delta q_k = \nu k^2 \delta q_k + \delta h_k + \int \mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}\,\Lambda_{jlk}q_j\delta q_k \tag{3.2}$$

$$\omega_k = \nu k^2 + \int \Lambda_{jlk}q_j\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}. \tag{3.3}$$

Another approach is obtained from the work of Kraichnan. Kraichnan has derived equations for the time dependent correlation functions, which are therefore more general in their scope than the present calculation. But the time response problem needs to be solved in his equation and to do this he uses the direct-interaction approximation which gives rise to coupled equations for the velocity correlation and the time decay of a fluctuation in the kth mode. As Kraichnan (1964) has pointed out, if one approximates his equations by forcing them into an exponential decay framework one can deduce (3.1), but ω_k is now to be taken as the eigenvalue of the first excited state of Liouville's equations treated as was (2.9). Within the present paper's framework this amounts to expanding the first state of Liouville's equation around $u_k P_0$ (which is the first excited state of $\mathscr{L}_0$). This gives

$$\omega_k = \nu k^2 + \int\int \frac{L_{kjl}q_j\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}}{\omega_j + \omega_l}. \tag{3.4}$$

This has also been deduced by Herring (1965) in his treatment of turbulence from a self-consistent field point of view. The choice of an exponential decay structure is also used in the quantum many-body problem in which it is known as the quasi-particle hypothesis (Edwards and Sherrington 1967, Balian and de Dominicis 1964). It is believed to have widespread validity, but the key problem of turbulence is to investigate the limit of very large Reynolds number which is equivalent to very strong coupling in the many-body problem, i.e. to critical behaviour, and there is no reason to expect the validity of the quasi-particle approximation there. The idea of ω_k being related to the time decay of the system has been further studied by Edwards (1965), but that treatment is not convincing. The basic trouble is that the equations contain integrations which are strongly divergent near $\boldsymbol{j} \sim 0$, so that in (3.1), for example, the two terms in the kernel are separately divergent and a finite answer can only arise if they cancel near $\boldsymbol{j} \sim 0$. But this means that near $\boldsymbol{j} \sim 0$ the modes are not really independent, i.e. when one talks of $\int\Lambda_{kjl}q_k q_j\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}$ as the flow out of the mode $\boldsymbol{k}$, and $\int\Lambda_{ljk}q_j q_l\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}$ as the flow into the mode $\boldsymbol{k}$, this becomes

rather meaningless since it is only the net change which has meaning. Indeed all the prescriptions mentioned above are in fact arbitrary, and from the present point of view any argument involving time scales is ruled out since the entire development has been for the steady state, and the problem of solving (2.18) lies entirely within that equation without reference to the fact that it is part of a more general problem. One needs a new principle to fix ω_k. In his original papers Kraichnan ennunciated a principle of maximal randomness, which he argued led to his equation for the response function. However, from the present point of view equations for the response function are, rather, additional hypotheses not directly related to a maximal property. The idea, however, that the state of turbulence is in some sense the most chaotic permitted by the equation of motion is an attractive one. One knows that in thermal equilibrium the system takes up all the phase space permitted to it with equal weight. It seems reasonable that the turbulent system will also occupy as much of phase space as is permitted by the equation of motion. In equilibrium special reasons are available to make the solution of the problem (the Gibbs distribution) easily accessible. In the present case there are no invariants available, but on the other hand there is in (2.10) a solution to the equations of motion. It is also true that the concept of entropy, when interpreted as information, is available for any system, without reference to equilibrium. The general form discussed by Shannon (Shannon and Weaver 1949) is

$$S = -\kappa \int P \ln P \tag{3.5}$$

where P is the probability distribution, and the integral is taken over all the variables of the distribution. It is to be noted that the value of S in general depends on the variables chosen. It is not possible to give a value for S independent of the choice of variables except in the case of thermal equilibrium which has canonical systems. Put in the sense that the entropy of a telephone exchange will depend on whether it is running in English or German this is not surprising, but at first sight it appears rather unpalatable in a physical theory. The fact is, however, that the accuracy of expansions given for q_k and that for S to be given below do depend on the variables used, and if instead of the u_k some other system were used an expansion of different accuracy would result. The u_k are however so obvious and useful as variables there seems no alternative or indeed the need for one. Since S defined by (3.5) is equivalent to that of the microcanonical ensemble, one can say that the expansion containing the least information, i.e. the maximum of entropy, will be given by

$$\frac{\partial S}{\partial \omega_k} = 0. \tag{3.6}$$

(The condition for a canonical ensemble would be $\delta F = 0$ where F is the free energy, since $F = F(T, V, N)$ where $T = \partial E/\partial S$. We prefer not to attempt to argue whether any analogue of F exists since we have $S = S(q_k, \omega_k)$ and with an equation for q_k in existence $\partial S/\partial \omega_k = 0$ will complete our set. It is clear that one would need a $T_k = \partial q_k/\partial S$ etc. and so an $F_k = q_k - T_k\, \partial S/\partial T_k$. In this sense S is the most fundamental variable, and one could even argue that our equation for q_k could be replaced by $\partial S/\partial q_k = 0$ or by putting in an external source A_k and differentiating twice with respect to A at $A = 0$. This latter turns out to be much the same as the form already used, so it will not be pursued.)

This equation is a direct maximization of the randomness of the system, and the question now arises as to its practicality. If one goes ahead naïvely, expanding $\ln P$ about P_0 one has firstly

$$\int P_0 \ln P_0 \prod_k \mathrm{d}u_k = \int \prod_k q_k^{-1/2} \exp\left(-\sum_k \frac{u_k u_{-k}}{q_k}\right) \times \left(\sum_k \frac{u_k u_{-k}}{q_k} - \tfrac{1}{2}\sum_k \ln q_k\right) \prod_k \mathrm{d}u_k \tag{3.7}$$

$$= \sum_k 1 - \tfrac{1}{2}\sum_k \ln q_k. \tag{3.8}$$

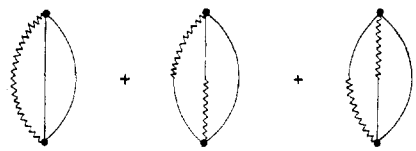
The $\Sigma_k 1$ is an infinite constant relating to the dimension of the $\boldsymbol{k}$ space and remains absolutely fixed throughout the calculation, so may be dropped. (It is of course simply the number of degrees of freedom of the system and quite uninteresting.) Hence

$$S_0 \to \frac{V}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{k}\,\ln q_k. \tag{3.9}$$

Collecting terms of order M^2 in the expansion of P the next terms are

$$\iiint \left(M_{kjl}M_{kjl}\frac{q_j q_l}{q_k} + M_{kjl}M_{lkj}q_j + M_{kjl}M_{ljk}q_l \right) \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}}{(\omega_k+\omega_j+\omega_l)^2} \tag{3.10}$$

other terms vanishing because of (2.27). These can be expressed in terms of diagrams



(3.11)

(which will be justified shortly). At this point we recall the remark made earlier in comparison of the present diagrams with the Feynman diagrams. The fact that physical quantities had to be extensive, i.e. depend on V alone, meant that only connected diagrams could occur, and, whereas the present diagrams were more complicated, these *topological* theorems had still to be valid. It is also to be noted that although in this paper our attention is on the extreme non-equilibrium system the mathematical structure is *topologically* no different from the quantum many-body problem. Now the entropy is also extensive and cluster expansions for it consist of all connected closed diagrams (whereas the correlation function is made of all connected diagrams with one line entering and one leaving). This has been shown in explicit detail for the quantum many-body problem, but in addition the method of splitting the lines has also been used in the many-body problem, and has been shown to be possible for the entropy by Balian and de Dominicis. Although, therefore, this is not physically the problem studied here we may be confident that a closed connected diagram expansion is available for the entropy. To obtain the expansion it is convenient to write

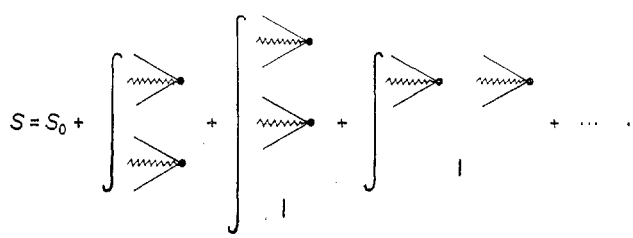
$$P = P_0(P_0^{-1}PP_0)$$

so that

$$S = S_0 + \int (P_0 + P_1 + \ldots)\ln\{1 + P_0^{-1}(P_1+P_2)P_0\}\delta u. \tag{3.12}$$

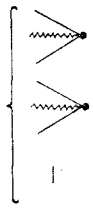
Keeping the principal line of the text for the expansion (2.25) of P, the powers from the logarithm can be written on lines of their own above that, i.e.

$$S = S_0 + \int_{P_1}^{P_0^{-1}P_1P_0} + \int_{P_0}^{P_0^{-1}P_1P_0} + \int_{P_0}^{P_0^{-1}P_2P_0} + \ldots \tag{3.13}$$

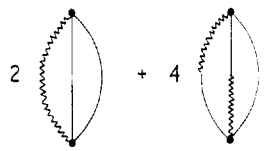


(3.14)

One now no longer has to think of the $\partial/\partial u$ line acting exlusively to the left; it can act in any direction since there is a P_0^{-1} on the left. The old rule still applies on the principal line which is the expansion of P. Thus from $(P_0^{-1}P_1P_0)^2P_0$, i.e.

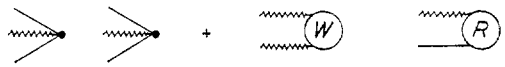


one obtains



(3.15)

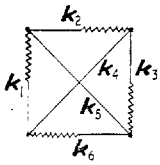
From $(P_0^{-1}P_1P_0)P_1$ one obtains the same. From P_2, i.e.



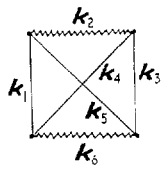
(3.16)

one obtains zero.

In general, diagrams will be quite complicated, e.g. from $(P_0^{-1}P_1P_0)^4$ one obtains (amongst others)



$$= \int \frac{MMMMq_{k_4}q_{k_5}\,\mathrm{d}^3(\boldsymbol{k}_1 \ldots \boldsymbol{k}_6)}{(\omega_2+\omega_3+\omega_4)(\omega_1+\omega_2+\omega_5)(\omega_3+\omega_5+\omega_6)(\omega_1+\omega_6+\omega_4)} \tag{3.17}$$



$$= \int \frac{MMMMq_1q_3q_4q_5q_2^{-1}q_6^{-1}\,\mathrm{d}^3(\boldsymbol{k}_1 \ldots \boldsymbol{k}_6)}{(\omega_2+\omega_3+\omega_4)(\omega_1+\omega_2+\omega_5)(\omega_3+\omega_5+\omega_6)(\omega_1+\omega_6+\omega_4)} \tag{3.18}$$

and so on. The most noticeable difference to the energy equation is the new occurrence of

i.e. q^{-1} in S: a formal proof that only connected diagrams survive will not be given since it has been considered by Balian and de Dominicis. Physically of course, since S must vary as V, this must be the case. If one differentiates S with respect to ω_k only, one obtains the desired second equation and so a pair of equations which are both nominal in M can be obtained.

4. Calculation

The equations are of course both of infinite order and one has to cut them off to make a closure. This must amount to cutting off at a certain power of M since the desirable properties of the equation hold to each order in M. When differentiating with respect to ω_k one has

$$\begin{aligned} q_k &= \frac{D_k}{\omega_k} \\ \frac{\delta q_j}{\delta \omega_k} &= \frac{q_k}{\omega_k}\delta(\boldsymbol{k}-\boldsymbol{j}) + \frac{1}{\omega_j}\frac{\partial D_j}{\partial \omega_k}. \end{aligned} \tag{4.1}$$

Strictly speaking one is therefore left with an unpleasant integral equation still to be solved for $\delta D_j/\delta\omega_k$ (which is obtained by differentiating the energy equation rewritten in terms of $D_k = q_k\omega_k$). In fact, in order to get an answer we shall drop this term; the computation involved in keeping it is tremendous and all we have been able to do is to check that the fact that a solution exists is not affected by the neglect of this term. One is then left, to order M^2, with the equations

$$\nu k^2 q_k + \int\int \frac{L_{lkj}q_k q_l - L_{kjl}q_j q_l}{(\omega_k+\omega_j+\omega_l)}\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l} = h_k \tag{4.2}$$

$$\frac{1}{2\omega_n} - \int \frac{L_{knl}q_n q_l - L_{lkn}q_k q_n}{q_k\omega_n(\omega_k+\omega_n+\omega_l)^2}\,\mathrm{d}^3\boldsymbol{l}\,\mathrm{d}^3\boldsymbol{k} - \int \frac{L_{kjn}q_n q_j\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{k}}{q_k\omega_n(\omega_k+\omega_j+\omega_n)^2}$$
$$+\int \frac{L_{njl}q_j q_l\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}}{\omega_n q_n(\omega_n+\omega_j+\omega_l)^2} - 2\left\{\int \frac{L_{njl}q_j q_l - L_{lnj}q_j}{(\omega_n+\omega_j+\omega_l)^3}\,\mathrm{d}^3\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{l}\right.$$
$$\left.+2\int \frac{L_{knl}q_n q_l - L_{lkn}q_n\,\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{l}}{(\omega_k+\omega_n+\omega_l)^3}\right\} = 0. \tag{4.3}$$

The case of a cascade of energy is that when h_k is peaked near the origin in $\boldsymbol{k}$, i.e. $h_k \simeq h\delta(\boldsymbol{k})$ and ν is rather small in the range of $\boldsymbol{k}$ of interest. Then energy enters very long wavelengths, cascades down the spectrum and is lost at large $\boldsymbol{k}$ (in a way which does not matter vitally to us, since there is so much $\boldsymbol{k}$ space as $|\boldsymbol{k}| \to \infty$ that the precise mechanism of loss does not matter). It has long been surmised that under these circumstances

$$q_k = |\boldsymbol{k}|^{-11/3}h^{2/3}q \tag{4.4}$$

$$q_k\,\mathrm{d}^3\boldsymbol{k} = \epsilon(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} = 4\pi|\boldsymbol{k}|^{-5/3}h^{2/3}q\,\mathrm{d}k \tag{4.5}$$

where q is a number; this is the Kolmogoroff distribution. As has been noted elsewhere (Edwards 1965), if it is assumed that $\omega_k = \omega h^{1/3}|\boldsymbol{k}|^{2/3}$ equation (4.2) does indeed have the solution (4.4), the integrals having to be evaluated very carefully to ensure the cancellation between the two terms of the kernel. The non-uniform convergence of the integrals near $\boldsymbol{k} = 0$ produces the $h\delta(\boldsymbol{k})$ of the right-hand side (and also another δ function at infinity which takes the place of $\nu k^2 q_k$ if the Kolmogoroff distribution is to hold over all $\boldsymbol{k}$ space). Now we may also see that $\omega_k = h^{1/3}|\boldsymbol{k}|^{2/3}\omega$; $q_k h^{2/3}|\boldsymbol{k}|^{-11/3}q$ is a solution of (4.3) provided that the integrals converge. Careful evaluation shows that they do, and that

$$\frac{2q}{\omega^2} = \frac{1}{4{\cdot}0\times\pi} \tag{4.6}$$

and, since from (4.2)

$$\frac{q^2}{\omega} = \frac{1}{1{\cdot}6\times\pi^2} \tag{4.7}$$

one has

$$\omega = 3{\cdot}3 \tag{4.8}$$

$$4\pi q = 5{\cdot}5. \tag{4.9}$$

The Kolmogoroff distribution is thus

$$\epsilon(|\boldsymbol{k}|) = 5{\cdot}5h^{2/3}|\boldsymbol{k}|^{-5/3}. \tag{4.10}$$

This is not a particularly good agreement with experiment (which according to Gibson and Schwartz (1963) is $1{\cdot}3\,h^{2/3}|\boldsymbol{k}|^{-5/3}$), but it does at least exist, and some rather crude approximations particularly in dropping the awkward terms in (4.1) have been made. One is of course making an expansion in (4.2), (4.3) which cannot be expressed in terms of a coupling constant, and in default of any watertight argument to justify it, the best thing has seemed to go ahead and see at least if sensible answers can be obtained. It is clear why (4.3) succeeds

compared with earlier attempts. The natural occurrence of pairs of terms in the kernel leads to vital cancellations lacking in, say, (3.3) which now appears to be a very arbitrary ansatz for closure. It can also be argued that since entropy can be defined for the non-equilibrium process, and since it seems reasonable that subject to the equations of motion the system will indeed take up a state of maximal randomness, the introduction of this function is a necessary and inevitable step in understanding these processes and in developing methods of solution.

5. Conclusion

It has been argued that the study of problems like turbulence which reduce to the solution of stochastic functional differential equations, is closely related to field theory and many-body problems. Thus techniques and functions developed for these problems in thermal equilibrium can be extended with appropriate modifications to the non-equilibrium solution of turbulence. The work in this paper is in many ways still of a tentative nature, and still lacks a criterion for the accuracy of the expansion. Nevertheless the theory achieves sufficient accuracy to obtain the Kolmogoroff spectrum and though to say that a theory is not manifestly false may appear small praise, it is still an advance, and it is hoped will lead to a better understanding of the phenomena and accuracy of expansion. We wish to emphasize again the lack of manoeuvrability in Hilbert space problems. We claim that for general operations one cannot go further than two unknown functions. It is of course always possible to introduce say $\langle u_k u_j u_l \rangle$, $\langle u_k u_j u_l u_m \rangle$ etc. as new unknowns and produce closures in terms of q_k and them. But these must be arbitrary in the sense firstly that they do not stem from general principles, as do (4.2) and (4.3), and that a general expansion cannot be made in terms of them.

Little has been said here about time dependence. Recent work of Kraichnan (1965) suggests that this cannot be studied by a simple extension of Eulerian concepts, and whether the present work will extend to time dependence remains to be seen.

Acknowledgments

The authors would like to thank Professor D. C. Leslie for many valuable discussions of this work, and for discovering some numerical errors in the first draft.

References

BALIAN, R., and DE DOMINICIS, C., 1964, *Physica*, **30**, 1927–32.
BERAN, M. J., 1968, *Statistical Continuum Theories* (New York: Interscience).
CHANDRASEKHAR, S., 1943, *Rev. Mod. Phys.*, **15**, 1.
EDWARDS, S. F., 1964, *J. Fluid Mech.*, **18**, 239.
—— 1965, *Int. Conf. on Plasma Physics, Trieste* (Vienna: International Atomic Energy Agency), p. 595.
EDWARDS, S. F., and SHERRINGTON, D., 1967, *Proc. Phys. Soc.*, **90**, 3.
GIBSON, C. H., and SCHWARTZ, W. H., 1963, *J. Fluid Mech.*, **16**, 365.
HASSELMANN, K., 1967, *Proc. R. Soc.*, **299**, 77.
HERRING, J., 1965, *Phys. Fluids*, **8**, 2219.
KRAICHAN, R. H., 1964, *Phys. Fluids*, **7**, 1163.
—— 1965, *Phys. Fluids*, **8**, 575.
PEIERLS, R. E., 1955, *Quantum Theory of Solids* (Oxford: Clarendon Press).
SHANNON, C. E., and WEAVER, W., 1949, *The Mathematical Theory of Communication* (Urbana, Ill.: University of Illinois Press).
WYLD, H., 1961, *Ann. Phys., N.Y.*, **14**, 143.